# Math 210C Lecture 28 Notes

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## 1 Induction-Restriction Tables, Determining Induced Characters, and Applications to Group Theory

### 1.1 Induction-restriction tables

**Definition 1.1.** Let $\psi_i$ be the irreducible characters of a group $H$, and let $\chi_i$ be the irreducible characters of $G$, where $H \leq G$. The **induction-restriction table** is the matrix with $(i, j)$-entry $\langle \psi_i, \mathrm{Res}_H^G(\chi_i) \rangle$ $(= \langle \mathrm{Ind}_H^G(\psi_i), \chi_j \rangle)$.

**Example 1.1.** Let $H = A_4$ and $G = S_4$. We have characters $1, \chi, \chi^2, \psi$ on $A_4$ and $\chi_1, \chi_2$, *dots*, $\chi_5$ on $S_4$. The induction-restriction table is

| | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\chi_4$ | $\chi_5$ |
|---|---|---|---|---|---|
| $1$ | 1 | 1 | 0 | 0 | 0 |
| $\chi$ | 0 | 0 | 1 | 0 | 0 |
| $\chi^2$ | 0 | 0 | 1 | 0 | 0 |
| $\psi$ | 0 | 0 | 0 | 1 | 1 |

What does this tell us? It tells us that $\mathrm{Ind}_H^G(1) = \chi_1 + \chi_2$, $\mathrm{Ind}_H^G(\chi) = \mathrm{Ind}_H^G(\chi^2) = \chi_3$, and $\mathrm{Ind}_H^G(\psi) = \chi_4 + \chi_5$.

### 1.2 Determining induced characters

**Proposition 1.1.** *Let $H \leq G$ have finite index, and let $g_1, \dots, g_k$ be coset representatives for $G/H$. Let $\psi$ be a character of $H$, and extend $\psi$ to $\tilde{\psi} : G \to F$ by $\tilde{\psi}(g) = 0$ if $g \in G \setminus H$. Then for $g \in G$,*

$$\mathrm{Ind}_H^G(\psi)(g) = \sum_{i=1}^{k} \tilde{\psi}(g_i^{-1} g g_i).$$

*Proof.* Let $W$ be a representation of $H$ with character $\psi$, and let $B = (w_1, \dots, w_n)$ be a basis. Then $\mathrm{Ind}_H^G(W) \cong F[G] \otimes_{F[H]} W$ has basis $g_i \otimes w_j$, where $g_i \in G$ and $w_j \in W$. We have $g \cdot g_i = g_{\sigma(i)} h_i$ for some $\sigma \in S_k$, $h_i \in H$ for all $1 \le i \le k$. Then

$$g(g_i \otimes w_j) = g_{\sigma(i)} h_i \otimes w_j = g_{\sigma(i)} \otimes h_i w_j.$$

Then $\rho_W(g)$, represented with respect to $\{g_i \otimes w_j\}$ (with lexicographical order), is a $k \times k$ matrix of $m \times m$ blocks ($m = \dim_F(W)$). Nonzero blocks $(i, \sigma(i)$ are the matrix representation of $\rho_W(h_i)$. Then

$$\mathrm{tr} = \sum_{\substack{i=1 \\ \sigma(i)=i}}^{k} \psi(h_i) = \sum_{\substack{i=1 \\ \sigma(i)=i}}^{k} \psi(g_i^{-1} h_i g_i) = \sum_{i=1}^{k} \tilde{\psi}(g_i^{-1} h_i g_i). \qquad \square$$

We don't have to only sum over representatives of cosets.

**Corollary 1.1.** *Let $G$ be finite and $H \le G$ have finite index. Then*

$$\mathrm{Ind}_H^G(g) = \frac{1}{|H|} \sum_{\substack{k \in G \\ kgk^{-1} \in H}} \psi(k^{-1} g k).$$

**Corollary 1.2.** *If $H \trianglelefteq G$ and $g \notin H$, then* $\mathrm{Ind}_H^G(g) = 0$.

**Example 1.2.** Let $G = D_{2n}$, where $|D_{2n}| = 4n$. Then $G^{\mathrm{ab}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so we get 4 abelian characters $\chi_1, \chi_2, \chi_3, \chi_4$ which are trivial, $s \mapsto 1$, $r \mapsto 1$, and $rs \mapsto 1$, respectively. Let $\psi : H = \langle r \rangle \to \mathbb{C}^\times$, where $\psi(r) = \zeta_n = e^{2\pi i/n}$. We then get characters of $H$, $1, \dots, \psi^{2n-1}$. If we let $\theta_i = \mathrm{Ind}_H^G(\psi^i)$, this vanishes on all reflections (by the previous corollary). The left coset representatives of $G/H$ are 1 and $s$, so $\theta_i(r^j) = \zeta_n^{ih} + \zeta_n^{-ij}$ from the proposition. The characters $\chi_i$ with $1 \le i \le n-1$ are distinct, have degree 2, and are irreducible (as not sums of abelian characters). We have 4 abelian characters, so the dimensions give $4 \cdot 1^1 + (n-1) \cdot 2^2 = 4n$. So these are all the characters of $G$.

Note that $\theta_i|_H = \psi^i + \psi^{-i}$ for $1 \le i \le n-1$, $\theta_0|_H = \chi_1 + \chi_3$, and $\theta_n|_H = \chi_2 + \chi_4$. So we can also write out the induction-restriction table.

**Corollary 1.3.** *Let $G$ be finite, and let $H \cap C_g = \coprod_{i=1}^{\ell} T_i$, where $T_i$ are distinct $H$-conjugacy classes. Let $h_i$ be a representative of $T_i$ for each $i$. Then*

$$\mathrm{Ind}_H^G(\psi)(g) = [G : H] \sum_{i=1}^{\ell} \frac{|T_i|}{|C_g|} \psi(h_i).$$

*Proof.* Note that $|\{k \in G : kgk^{-1} \in T_i\}| = |Z_g| \cdot |T_i| = \frac{n}{|C_g|}|T_i|$. By the first corollary,

$$\mathrm{Ind}_H^G(\psi)(g) = \frac{1}{|H|} \sum_{i=1}^{\ell} \frac{n}{|C_g|} |T_i| \psi(h_i). \qquad \square$$

**Example 1.3.** Let $H = S_3 \leq G = S_4$. Let $C_1, \ldots, C_5$ be the conjugacy classes in $S_4$ of $e, (1\,2), (1\,2\,3), (1\,2\,3\,4), (1\,2)(3\,4)$ with $c_i = |C_i|$. Let $T_1, T_2, T_3$ be the conjugacy classes in $S_3$ of $e, (1\,2), (1\,2\,3)$. Then $C_4 \cap S_3 = C_5 \cap S_3 = 0$, and $C_i \cap S_3 = T_i$ for al l$1 \leq i \leq 3$. If we let $\psi_1 = 1$, $\psi_2 = \mathrm{sgn}$, $\psi_3 = \chi_W$ of $S_4$, then let $\phi_i = \mathrm{Ind}_H^G(\psi_i)$. We can then calculate these characters of $S_4$:

| $S_4$ | $e$ | $(1\,2)$ | $(1\,2\,3)$ | $(1\,2\,3\,4)$ | $(1\,2)(3\,4)$ |
|---|---|---|---|---|---|
| $\phi_1$ | 4 | 2 | 1 | 0 | 0 |
| $\phi_2$ | 4 | $-2$ | 1 | 0 | 0 |
| $\phi_3$ | 8 | 0 | 1 | 0 | 0 |

Then we can calculate

$$\langle \phi_i, \phi_i \rangle = \frac{1}{24}|\phi_1(e)|^2 + \frac{1}{4}|\phi_i((1\,2))|^2 + \frac{1}{3}|\phi_i((1\,2\,3))|^2 = \begin{cases} 2 & i = 1, 2 \\ 3 & i = 3. \end{cases}$$

We also have $\langle \phi_1, \chi_1 \rangle = 1$ and $\langle \phi_1, \chi_2 \rangle = 0$, so $\phi_1 - \chi_1$ irreducible of degree 3; this is what we call $\chi_4$. Similarly, $\phi_2 - \chi_2$ is irreducible of degree 3 and is what we would call $\chi_5$. We can also calculate

$$\langle \phi_3, \chi_4 \rangle = \langle \chi_3, \chi_5 \rangle = 1,$$

which gives us that $\phi_3 - \chi_4 - \chi_5$ is irreducible of degree 2. This is what we would call $\chi_3$. So we can find all the irreducible characters in this way.

### 1.3 Properties of groups in terms of characters

Let $G$ be finite with irreducible characters $\chi_1, \ldots, \chi_r$. We can calculate properties of the group from the character table.

**Proposition 1.2.** *$N_i = \{g \in G : \chi_i(g) = \chi_i(1)\} \trianglelefteq G$, and all normal subgroups of $G$ have the form $\bigcap_{i \in J} N_i$ for $J \subseteq \{1, \ldots, r\}$.*

**Proposition 1.3.** *$Z_i = \{g \in G : |\chi_i(g)| = \chi_i(1)\} \trianglelefteq G$, and $\bigcap_{i=1}^r Z_i = Z(G)$.*

**Proposition 1.4.** *Let $1 \leq i, j \leq r$, let $c_j = |C_j|$, let $n_i = \deg(\chi_i)$, and let $g \in G$. Then $\frac{c_j}{n_i}\chi_i(g_j) \in \mathbb{Z}[\mu_n]$.*

**Corollary 1.4.** *Let $n_i$ be the degree of an irreducible representation. Then $n_i \mid n$.*

You can use this to prove the following important theorem.

**Theorem 1.1** (Burnside's theorem)**.** *Every group of order $p^a q^b$ with $p, q$ prime is solvable.*

Hopefully, you are now an expert in abstract algebra.